Solutions: Homework 5

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Problem 1. Let T be a Möbius transformation with fixed points z_1 and z_2. If S is a Möbius transformation show that $S^{-1}TS$ has fixed points $S^{-1}z_1$ and $S^{-1}z_2$.

Proof. T is not the identity map, hence $S^{-1}TS$ is also not the identity map. This means that $S^{-1}TS$ has at most two fixed points. Now, $S^{-1}TS(S^{-1}z_1) = S^{-1}T(z_1) = S^{-1}(z_1)$. Therefore, $S^{-1}(z_1)$ is a fixed point of $S^{-1}TS$. Similar calculations show that $S^{-1}z_2$ is also a fixed point of $S^{-1}TS$. Also $S^{-1}z_1 \neq S^{-1}z_2$. So $S^{-1}TS$ has two fixed points, $S^{-1}z_1$ and $S^{-1}z_2$. □

Problem 2. (a) Show that a Möbius transformation has 0 and ∞ as its only fixed points iff it is a dilation.
(b) Show that a Möbius transformation has ∞ as its only fixed point iff it is a translation.

Proof. (a) Suppose S is a dilation. Then $S(z) = az$ for some $a \neq 0, 1$. Then S in not the identity and $S(0) = 0$ and $S(\infty) = \infty$, and thus these are the only fixed points of S. Conversely, suppose that 0 and ∞ are the only fixed points of a Möbius transformation $S(z) = \frac{az+b}{cz+d}$. Since $S(0) = 0, b = 0$. Since $S(\infty) = \infty, c = 0$. So, $S(z) = (ad^{-1})z$ and S is not the identity, and hence it is a dilation.
(b) Suppose S is a translation. Then $S(z) = z + a$ for some $a \neq 0$. Clearly S has no fixed points in $\mathbb{C}$ and $S(\infty) = \infty$. Conversely, suppose that ∞ is the only fixed point of a Möbius transformation $S(z) = \frac{az+b}{cz+d}$. Since $S(\infty) = \infty, c = 0$. Now suppose that $a \neq d$. Then $S(\frac{b}{d-a}) = \frac{b}{d-a}$, which contradicts the fact that ∞ is the only fixed point of S. So $a = d$ and $S(z) = z + bd^{-1}$ and S is not the identity, and hence it is a translation. □

Problem 3. Show that a Möbius transformation T satisfies $T(0) = \infty$ and $T(\infty) = 0$ iff $T(z) = az^{-1}$ for some $a \in \mathbb{C}$.

Proof. Suppose $T(z) = az^{-1}$ for some $a \in \mathbb{C} \setminus \{0\}$. Then $T(0) = \infty$ and $T(\infty) = 0$. Conversely, suppose that $T(0) = \infty$ and $T(\infty) = 0$ and $T(z) = \frac{az+b}{cz+d}$. Since $T(0) = \infty, d = 0$. Since $T(\infty) = 0, a = 0$. So, $T(z) = (bc^{-1})z^{-1}$. □

Problem 4. Prove the following analogue of Leibniz's rule. Let G be an open set and let γ be a rectifiable curve in G. Suppose that $\phi : \{\gamma\} \times G \to \mathbb{C}$ is a continuous function and define $g : G \to \mathbb{C}$ by

$$g(z) = \int_\gamma \phi(w, z)dw$$

then g is continuous. If $\frac{\partial\phi}{\partial z}$ exists for each (w,z) in $\{\gamma\}\times G$ and is continuous then g is analytic and

$$g'(z)=\int_\gamma \frac{\partial\phi}{\partial z}(w,z)dw.$$

Proof. Let $z\in G$. Since continuity is a local property, we can just work in a small open ball around z. Choose a closed ball V around z contained in this open ball. Then V is compact. Since a finite product of compact sets is compact, $\{\gamma\}\times V$ is compact, hence $\phi|_V:\{\gamma\}\times V\to\mathbb{C}$ is uniformly continuous. So, given $\epsilon>0$, there exists $\delta>0$ such that $|\phi(w_1,z_1)-\phi(w_2,z_2)|<\epsilon$ whenever $|w_1-w_2|<\delta$ and $|z_1-z_2|<\delta$ with $(w_1,z_1),(w_2,z_2)\in\{\gamma\}\times V$. Now, let h be small such that $z+h\in V$ and $|h|<\delta$. We can do this because any closed ball contains an open ball. Then

$$|g(z+h)-g(z)|=\left|\int_\gamma \phi(w,z+h)-\phi(w,z)dw\right|<\epsilon V(\gamma)$$

where $V(\gamma)$ denotes the total variation of γ, which is a finite number as γ is rectifiable. This shows that g is continuous at z. Since z is arbitrary and G is open, this shows that g is continuous everywhere.
Now suppose that $\frac{\partial\phi}{\partial z}$ exists for each (w,z) in $\{\gamma\}\times G$ and is continuous. Then $\frac{\partial\phi}{\partial z}|_V:\{\gamma\}\times V\to\mathbb{C}$ is uniformly continuous. Let us denote by ϕ_1 the function $\frac{\partial\phi}{\partial z}|_V$. Given $\epsilon>0$, there exists $\delta>0$ such that $|\phi_1(w_1,z_1)-\phi_1(w_2,z_2)|<\epsilon$ whenever $|w_1-w_2|<\delta$ and $|z_1-z_2|<\delta$ with $(w_1,z_1),(w_2,z_2)\in\{\gamma\}\times V$. Let h be small such that $z+h\in V$ and $|h|<\delta$. Then

$$\left|\int_0^1 \phi_1(w,z+th)-\phi_1(w,z)dt\right|\le\int_0^1|\phi_1(w,z+th)-\phi_1(w,z)|dt<\epsilon$$

for any $w\in\{\gamma\}$. Note that $\frac{d}{dt}\left(\frac{\phi(w,z+th)}{h}\right)=\phi_1(w,z+th)$. So

$$\int_0^1 \phi_1(w,z+th)-\phi_1(w,z)dt=\frac{\phi(w,z+h)-\phi(w,z)}{h}-\phi_1(w,z).$$

Therefore,

$$\left|\frac{\phi(w,z+h)-\phi(w,z)}{h}-\phi_1(w,z)\right|<\epsilon$$

for all $w\in\{\gamma\}$ and $|h|<\delta$ with $z+h\in V$. Now,

$$\left|\frac{g(z+h)-g(z)}{h}-\int_\gamma\frac{\partial\phi}{\partial z}(w,z)dw\right|=\left|\int_\gamma\frac{\phi(w,z+h)-\phi(w,z)}{h}-\frac{\partial\phi}{\partial z}(w,z)dw\right|<\epsilon V(\gamma)$$

So g is differentiable at z with

$$g'(z)=\int_\gamma \frac{\partial\phi}{\partial z}(w,z)dw.$$

Since $\frac{\partial\phi}{\partial z}$ is continuous, g' should also be continuous by the first part of this question. So, g is analytic. □

Problem 5. Suppose that γ is a rectifiable curve in $\mathbb{C}$ and ϕ is defined and continuous on $\{\gamma\}$. Use the above exercise to show that

$$g(z) = \int_\gamma \frac{\phi(w)}{w-z}dw$$

is analytic on $\mathbb{C} \setminus \{\gamma\}$ and

$$g^{(n)}(z) = n! \int_\gamma \frac{\phi(w)}{(w-z)^{n+1}}dw.$$

Proof. Let $G = \mathbb{C} \setminus \{\gamma\}$. G is open in $\mathbb{C}$. Let $f : \{\gamma\} \times G \to \mathbb{C}$ denote the function defined by $f(w, z) = \frac{\phi(w)}{w-z}$. f is continuous as ϕ is continuous. Then by Problem 4, g is analytic on $G = \mathbb{C} \setminus \{\gamma\}$ and

$$g'(z) = \int_\gamma \frac{\phi(w)}{(w-z)^2}dw.$$

Now repeating the argument for $f_1 = \frac{\phi(w)}{(w-z)^2}$ defined on $\{\gamma\} \times G$, we get

$$g''(z) = 2 \int_\gamma \frac{\phi(w)}{(w-z)^3}dw.$$

Repeating the argument n times gives us

$$g^{(n)}(z) = n! \int_\gamma \frac{\phi(w)}{(w-z)^{n+1}}dw.$$

$\square$